PD Tracking for a Class of Underactuated Robotic Systems With Kinetic Symmetry

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***Abstract*— In this letter, we study stability properties of Proportional-Derivative (PD) controlled underactuated robotic systems for trajectory tracking applications. Stability of PD control laws for fully actuated systems is an established result, and we extend it for the class of underactuated robotic systems. We will first show some well known examples where PD tracking control laws do not yield tracking; some of which can even lead to instability. We will then show that for a subclass of robotic systems, PD tracking control laws, indeed, yield desirable tracking guarantees. We will show that for a specified time interval, and for sufficiently large enough PD gains (input saturations permitting), local boundedness of the tracking error can be guaranteed. In addition, for a class of systems with the kinetic symmetry property, stronger conditions like convergence to desirable bounds can be guaranteed. This class is not restrictive and includes robots like the acrobot, the cart-pole, and the inertia-wheel pendulums. Towards the end, we will provide necessary simulation results in support of the theoretical guarantees presented.**

***Index Terms*— PD control, underactuated robotics, kinetic symmetry.**

I. Introduction

Underactuated robotic systems are systems with fewer inputs than degrees of freedom. Control or stabilization of this class of systems is hard, due to the presence of zero dynamics [1], [2]; unstable zero dynamics can lead to instability of the entire system. Some of the methods used to control this class of systems are partial feedback linearization [3], nested saturations [4], and energy methods [5]. However, a lot of these methods are nonlinear and heavily dependent on the model, thereby, failing to gain acceptance in the industry. This is reaffirmed by a recent industry survey [6], which showed that the most popular control law used, even today, is the PD (or even PID) control.

Given the high acceptance rating of PD tracking control laws, it is worthwhile to analyze their stability guarantees. For the class of fully actuated robotic systems, there are a slew of convergence guarantees: asymptotic convergence is established in [7], exponential stability for a constant desired configuration is established in [8], [9], and ultimate boundedness for a time varying desired configuration is established in [10], [11]. On the contrary, only preliminary results exist for the class of underactuated robotic systems. For example, in the classic cart-pole system, it can be shown that local stability results can be achieved as long as the pole is above the horizontal [12]. This is established via linearization about its vertical position. However, stability guarantees for systems with time varying desired trajectories cannot be provided via linearization. Similarly, there are preliminary results of stability for PD controlled bipedal walking robots [13]. Local convergence and boundedness was established in [13] by imposing assumptions on the zero dynamics i.e., the zero dynamics is assumed to have a stable periodic orbit. However, for the class of continuous robotic systems, these assumptions are restrictive. In a lot of the applications, the goal is to provide reasonable tracking guarantees regardless of the zero dynamics being stable or not. Therefore, there is a need for a detailed study on the set of conditions, with which convergence guarantees for PD control laws can be provided.

The goal in this letter is to identify the types of underactuated robotic systems, and the associated set of assumptions, for which tracking guarantees can be provided for PD control laws. The results presented are mainly motivated by [13]. In particular, local convergence to a bound in a finite interval was established in [13, Lemma 1], which was then utilized to establish stability of the full system. This was achieved for the class of robots with bounded inertia matrices (known as class $\mathcal{BD}$ [14]). The letter generalizes this result to include a larger class of systems i.e., we include tracking in the unactuated joint coordinates as well. With this result, we then establish the main result of the letter for a sub-class of robotic systems i.e., systems where the kinetic energy is invariant of some of the configuration variables (known as kinetic symmetry [2]). In particular, we show that for any specified time interval, exponential convergence of the error to a desirable bound can be established for that interval. This will be validated by simulating two underactuated robot models: the acrobot, and the cart-pole (shown by Fig. 1).

The paper is structured as follows: Section II will provide a description of the robot model. We will describe the various types of underactuations and the associated model properties. Section III will describe the PD control law. We will provide concrete examples of some underactuated robot models, where PD control fails. Equipped with this analysis, we will then identify conditions that, when satisfied, eliminate the failure cases. These conditions are not restrictive, and, in fact, include a large body of underactuated systems in practice. For this

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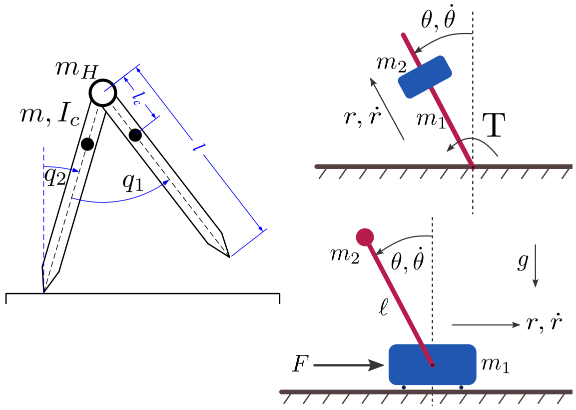

Fig. 1. Figure showing some examples of underactuated systems: the acrobot (left), the pendulum-slider (top right), and the cart-pole system (bottom right).

class of systems, we present the main results in Section IV, and the simulation results in Section V.

Notation. Let $\mathbb{R}$ denote the set of real numbers, and $\mathbb{R}^n$ denote the Euclidean space of dimension n. An open Euclidean ball of radius $r > 0$ centered at $x \in \mathbb{R}^n$ is denoted by $\mathbb{B}_r(x)$. For any $x \in \mathbb{R}^n$, the Euclidean norm is denoted by $|x|$. Given a symmetric matrix $A \in \mathbb{R}^{m\times m}$, we denote its minimum and maximum eigenvalues as $\lambda_{\min}(A), \lambda_{\max}(A)$ respectively. Norm of A is denoted by $\|A\|$.

II. Robot model

We consider an n-DOF rigid robotic system, with the configuration manifold Q. The state is denoted by $x := (q, \dot q) \in T\mathrm{Q}$, which is of dimension $2n$, and the torque input is denoted by $u \in \mathbb{R}^m$, which is of dimension $m < n$. Given the states and inputs, the Euler-Lagrangian dynamics is given by the following:

$$D(q)\ddot q + C(q,\dot q)\dot q + G(q) = Bu, \tag{1}$$

where $D(q) \in \mathbb{R}^{n\times n}$ is the inertia matrix, $C(q,\dot q) \in \mathbb{R}^{n\times n}$ is the Coriolis-centrifugal matrix, $G(q) \in \mathbb{R}^n$ is the gravity vector, and $B \in \mathbb{R}^{n\times m}$ is the mapping of the torques to the joints. (1) is obtained from the Lagrangian $\mathcal{L}(q,\dot q) := \frac{1}{2}\dot q^T D(q)\dot q - \mathcal{V}(q)$, where $\mathcal{V} : \mathrm{Q} \to \mathbb{R}$ is the potential energy. Specifically the left hand side (LHS) of (1) is obtained as

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot q_i} - \frac{\partial \mathcal{L}}{\partial q_i}, \quad i = 1, 2, \ldots, n. \tag{2}$$

In this letter, we will specifically focus on systems with non-interacting inputs i.e., each joint, if actuated, is independently and directly controlled by an actuator. Hence, unlike the matrices D, C, G, it is assumed that B is well known[1]. Hence B will be a tall matrix with the rows corresponding to the unactuated coordinate being identically zero. The remaining rows will consist of only one element with value 1. Note that D, C have some important properties, namely, D is symmetric positive definite, and $\dot D - 2C$ is skew-symmetric [14]. These properties will be useful in the next section. (1) can be represented in statespace form as

$$\dot x = f(x) + g(x)u, \tag{3}$$

by appropriate determination of f, g. Since the robot is under-actuated ($m < n$), we can rearrange the rows and identify two types of configuration variables: a) the shape variables q^m of dimension m, which are to track a specific desired trajectory, and b) the external variables q^z of dimension $l = n - m$, which are the remaining elements of q. Accordingly, we can separate the dynamics of the robot (1) into two parts

$$\begin{aligned} D_{11}(q)\ddot q^z + D_{12}(q)\ddot q^m + C_1(q,\dot q)\dot q + G_1(q) &= B_1 u \\ D_{21}(q)\ddot q^z + D_{22}(q)\ddot q^m + C_2(q,\dot q)\dot q + G_2(q) &= B_2 u. \end{aligned} \tag{4}$$

The terms corresponding to D, C, G and B are apparent from the setup and their explicit expressions are avoided in the interest of space. If q^z is unactuated, then $B_1 = 0$. Therefore $\ddot q^z$ can be eliminated from (4) to obtain

$$\begin{aligned} (D_{22} - D_{21}D_{11}^{-1}D_{12})\ddot q^m + &(C_2 - D_{21}D_{11}^{-1}C_1)\dot q \\ &+ G_2 - D_{21}D_{11}^{-1}G_1 = B_2 u. \end{aligned} \tag{5}$$

Alternatively, there are classes of systems where q^m is unactuated, for which $B_2 = 0$. Accordingly, we obtain $-D_{21}D_{11}^{-1}B_1 u$ on the right hand side (RHS) of (5). In this case, D_{21} must be non-zero for u to have any effect on $\ddot q^m$.

In a similar fashion, other types of combinations i.e., tracking of a mix of actuated and unactuated coordinates are also possible, and their formulations will be similar to (5). Hence, as long as the number of joints to be tracked is equal to the number of inputs, we can represent the following reduced dynamics:

$$D_u(q)\ddot q^m + C_u(q,\dot q)\dot q + G_u(q) = B_u(q)u, \tag{6}$$

where q^m is the configuration to be tracked, D_u, C_u, G_u are given by (5), and $B_u : \mathrm{Q} \to \mathbb{R}^{m\times m}$ is the mapping matrix appropriately obtained. Similar to D, D_u is also symmetric positive definite (see [13, Proposition 1]). Specific restrictions on B_u will be imposed in the next section. Depending on the mapping matrix B, the shape variable can be actuated, unactuated or partially actuated, and the goal is to study tracking performances when a PD control law is applied.

Systems with kinetic symmetry. In this letter, we are particularly interested in a sub-class of robotic systems, where the kinetic energy is invariant of some of the configuration coordinates. For example, if the system has kinetic symmetry w.r.t. the i^{th} coordinate q_i, then $\frac{\partial \dot q^T D(q)\dot q}{\partial q_i} = 0$. In this manuscript, we will be interested in systems having kinetic symmetry w.r.t. the external variables q^z. A large class of systems such as walking robots, cart-pole systems, and serial chain manipulators fall in this category. These systems have salient properties that allow us to establish stronger results for PD based control laws. For example, we can infer from (2) that

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot q^z} = -\frac{\partial \mathcal{V}(q)}{\partial q^z}, \tag{7}$$

which is purely a function of q. This will be used in the results that follow.

[1] Later on in Assumption 2, additional restrictions will be imposed on the types of allowable B, which will be useful for simplifying the results.

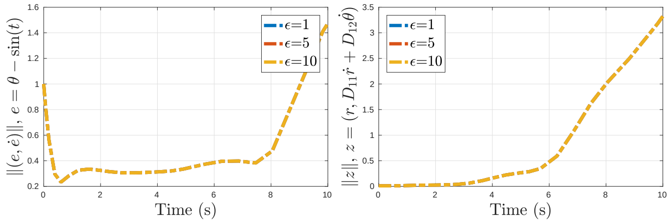

Fig. 2. Figures showing the PD tracking results as a function of time for the pendulum-slider system. All positions are in radians. See Fig. 1 for the pictorial representation of this system. Gravity is ignored for convenience.

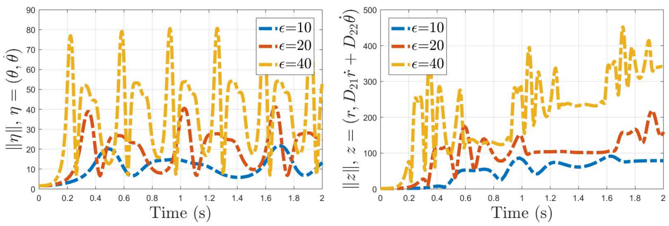

Fig. 3. Figures showing the tracking results for the cart-pole system. Position errors are measured in radians. See Fig. 1 for the pictorial representation of this system. Note that $\boldsymbol{\eta}=(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})$.

III. PD TRACKING WITH UNDERACTUATION

Having obtained the underactuated robot model, we are now ready to study PD tracking for these types of systems. For q^m to be tracked, we have the desired trajectory $q_d : \mathbb{R}_{\geq 0} \to \mathcal{P}(\mathrm{Q})$, where $\mathcal{P}$ is the canonical projection for q^m. We would like to obtain suitable PD gains that yield tracking of this desired trajectory. Therefore, we define the following relative degree two output:

$$e(t, q^m) = q^m - q_d(t), \tag{8}$$

where e is the error between the actual and the desired values. For convenience, we will impose the following assumption on the desired trajectory:

Assumption 1: *For the class of robot manipulators $\mathcal{BD}$, the desired configuration $q_d : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ is chosen such that it is two times differentiable, and its first and second derivatives are uniformly bounded by some $c_q > 0$.*

Note that this is not a restrictive assumption, as the actuators have practical limits (both speed and torque). With this desired trajectory, we use the following PD control law:

$$u_{\mathrm{PD}}(t, q^m, \dot{q}^m) = -K_p(q^m - q_d(t)) - K_d(\dot{q}^m - \dot{q}_d(t)), \tag{9}$$

where K_p, K_d are the gain matrices of dimension m. For simplicity, we will assume that equal gains are applied for every joint i.e., $K_p = k_p\mathbf{1}$, $K_d = k_d\mathbf{1}$ for some k_p, $k_d > 1$ and an identity matrix $\mathbf{1}$ of appropriate size. Having defined this PD control law (9), we have the resulting closed loop dynamics of (3) as

$$\dot{x} = f^{cl}(t, x) := f(x) + g(x)u_{\mathrm{PD}}(t, q^m, \dot{q}^m). \tag{10}$$

Two types of questions can be asked about the tracking/stability performance of this closed loop system (10): a) can we provide tracking/stability guarantees for all classes of robotic systems? (b) can we guarantee convergence from any initial state on Q? To answer these questions, we will present two concrete examples to show that tracking can fail under more than one situations.

Example 1: *Consider a 2-DOF pendulum-slider system i.e., a pendulum is hinged on one side with a prismatic joint sliding on the top. The pendulum is actuated, while the slider is not. See Fig. 1 for more details. For simplicity, we will not include gravity terms for this example. We will choose the desired trajectory for the pendulum to be a trigonometric function: $q_d(t) = \sin(t)$. The results for applying the PD control law with $k_p = \varepsilon^2$, $k_d = 2\varepsilon$ are shown in Fig. 2. It can be observed that the tracking error is increasing over time despite increasing the gains. This is because the slider position r is increasing in an unbounded fashion for all t, thereby affecting the overall rotational inertia of the pendulum about its pivot.*

It is worth noting that the class of manipulators used in practice rarely have the type of configuration shown in Example 1. To illustrate, walking robots like AMBER [15], and manipulators like SCARA [16] do not have this problem. Therefore, the first step in this letter is to choose a sub-class of manipulators called the class $\mathcal{BD}$ [14] manipulators. This class can have any one of the following configurations:

(a) All joints are prismatic.
(b) All joints are revolute.
(c) A series of prismatic joints followed by a series of revolute joints.
(d) Configurations where the axis of translation of each prismatic joint is parallel to all preceding revolute joints.

Class $\mathcal{BD}$ manipulators have desirable properties, which are useful for establishing stability guarantees:

Property 1: *For the class of manipulators $\mathcal{BD}$ there exist $c_l, c_u > 0$ such that $\forall\ (q, \dot{q}) \in T\mathrm{Q}$,*

(a) $c_l \leq \|D(q)\| \leq c_u$
(b) $c_l \leq \|D^{-1}(q)\| \leq c_u$
(c) $\|\dot{D}(q, \dot{q})\| \leq c_u|\dot{q}|$
(d) $\|C(q, \dot{q})\| \leq c_u|\dot{q}|$
(e) $\|G(q)\| \leq c_u$

Property 1 is well established in literature, and can be found in [14] (for D), [17] (for C), and in [18] (for G). Accordingly, D_u, C_u, G_u have the following properties:

Property 2: *For the class of robot manipulators $\mathcal{BD}$ there exist positive constants c_l, c_u such that $\forall\ (q, \dot{q}) \in T\mathrm{Q}$,*

(a) $c_l \leq \|D_u(q)\| \leq c_u$
(b) $c_l \leq \|D_u^{-1}(q)\| \leq c_u$
(c) $\|\dot{D}_u(q, \dot{q})\| \leq c_u|\dot{q}|$
(d) $\|C_u(q, \dot{q})\| \leq c_u|\dot{q}|$
(e) $\|G_u(q)\| \leq c_u$

Note that we have used the same constants c_l, c_u for ease of notations. Proof of Property 2 is in [13, Appendix A]. Despite restricting our study to the class of $\mathcal{BD}$ manipulators, the following example shows that PD tracking can still fail:

Example 2: *Consider a cart-pole system shown in Fig. 1. The cart is actuated, but the pendulum is unactuated and free to rotate in any direction. We will choose the desired trajectory for the pendulum to be, $q_d(t) = 2\sin(t)$, which has a higher amplitude. The results are shown in Fig. 3 with the proportional gain $k_p = \varepsilon^2$, and the derivative gain $k_d = 2\varepsilon$. It was observed that irrespective of the gains applied, the tracking failed when the pendulum angle crossed $\pi/2$.*

In Example 2, the choice of the actuated/unactuated coordinate was affecting the tracking performance. At $\theta = \pi/2$, $B_u = 0$, resulting in a non-inertially coupled configuration [3], thereby resulting in poor tracking. However, local results can still be achieved by choosing a subset of the configuration space $\mathrm{Q}_u \subset \mathrm{Q}$, with $q_d(t) \in \mathcal{P}(\mathrm{Q}_u)$. Furthermore, in order to simplify the results that follow, we will impose a stronger restriction on the allowable B_u:

Assumption 2: *For the class of robot manipulators $\mathcal{BD}$, q_d, Q_u are chosen such that for a small enough $\iota > 0$, and $\forall$ $t \geq 0$, $q \in \mathrm{Q}_u$,*

$$\Lambda := \begin{bmatrix} B_u + B_u^T & B_u + (1+|e|)(B_u^T - \iota\mathbf{1}) \\ B_u^T + (1+|e|)(B_u - \iota\mathbf{1}) & (1+|e|)(B_u + B_u^T) \end{bmatrix}$$

is symmetric positive definite.

Note that this is not a restrictive assumption. For example, for systems where the actuated configurations are required to track a trajectory, the mapping B_u becomes an identity (e.g., acrobot), thereby naturally satisfying Assumption 2. Similar observations can be made for other types of configurations.

IV. Main results

We are now ready to study the stability properties of underactuated robotic systems when a PD control law is applied. We will first study the zero dynamics that is associated with underactuations, and then present the main results.

Normal forms. It is well known that for underactuated mechanical systems, there exists a global change of coordinates that yields two sets of equations that correspond to the controlled and uncontrolled dynamics respectively [1]. Hence, for the given set of shape variables q^m, we have the corresponding change of coordinates: $\Phi_t : T\mathrm{Q}_u \to \mathbb{R}^{2n}$ that yields $\Phi_t(q, \dot{q}) := (e(t, q^m), \dot{e}(t, q^m, \dot{q}^m), z(q, \dot{q}))$, where $z : T\mathrm{Q}_u \to \mathbb{R}^{2(n-m)}$ are the new uncontrolled states. Accordingly, the statespace dynamics can be expressed via $e, \dot{e}, z$. If the tracking error is zero, then the resulting dynamics of z is called the zero dynamics [1]. The zero dynamics lies on

$$\mathbb{Z}_t = \{(q, \dot{q}) \in T\mathrm{Q}_u : e(t, q^m) = 0, \dot{e}(t, q^m, \dot{q}^m) = 0\}. \quad (11)$$

We will assume that the solution of the zero dynamics is forward complete [19].

It is worth noting that even if the diffeomorphism is guaranteed to exist, explicit analytical expressions are difficult to find. However, for systems with kinetic symmetry, this can be analytically obtained. This sub-class of systems will be discussed later on. We will first present a more general result, which is a straightforward extension of the results presented in [13]. For the initial state $(q_0, \dot{q}_0) \in \mathrm{Q}_u$, we will denote the evolution of the error as a function of time by $(e(t), \dot{e}(t))$, with $e(0) = q_0^m - q_d(0)$, $\dot{e}(0) = \dot{q}_0^m - \dot{q}_d(0)$. Similarly, let $z_0 \in \mathbb{Z}_0$ (by a slight abuse of notation) be the initial state of z, and let $z^*(t)$ be the solution for the zero dynamics for all $t \geq 0$. We then have the following theorem:

Theorem 1: *Given the class of manipulators $\mathcal{BD}$, let the configuration set Q_u, and the desired configuration q_d be picked such that Assumptions 1 and 2 are satisfied. Then we have the following:*

(a) *For every $(q_0, \dot{q}_0) \in T\mathrm{Q}_u$, there exist sufficiently large enough gains $k_p, k_d > 1$, and a correspondingly small enough $T_\delta > 0$ such that the outputs $(e(t), \dot{e}(t))$ are exponentially convergent to a bound for all $t \in [0, T_\delta]$.*

(b) *For every $T > 0$, and for every $z(0) \in \mathbb{Z}_0$, there exists a small enough $r > 0$, and sufficiently large enough gains $k_p, k_d > 1$ such that for all $(e(0), \dot{e}(0)) \in \mathbb{B}_r(0, 0)$, the outputs $(e(t), \dot{e}(t))$ are exponentially convergent to a bound for all $t \in [0, T]$.*

Remark 1: When we say that the error is exponentially convergent to a bound in the interval $[0, T_\delta]$ we mean that there exist $M, \lambda, d > 0$ such that

$$|(e(t), \dot{e}(t))| \leq M e^{-\lambda t} |(e(0), \dot{e}(0))| + d, \forall t \in [0, T_\delta]. \quad (12)$$

Both (a) and (b) of Theorem 1 are very similar, except that the roles of some of the variables are reversed. In particular, (a) establishes that for every initial state there is a small enough interval $[0, T_\delta]$, in which the boundedness is ensured, and similarly (b) establishes that for every closed interval there is a small enough neighborhood for $(e(0), \dot{e}(0))$, in which the boundedness is ensured. Note that this theorem is an extension of [13, Lemma 1 and Lemma 2]. Therefore, proofs of both the parts follow the steps in [13, Proofs of Lemma 1 and Lemma 2] respectively. In particular, the derivative of the Lyapunov candidate chosen will now consist of the time dependent $\dot{q}_d, \ddot{q}_d$, which are replaced by their bounds (by Assumption 1). In this letter, we will prove a stronger result for a sub-class of robotic systems that have kinetic symmetry.

Systems with kinetic symmetry. With the presence of kinetic symmetry, there are explicit forms for the change of coordinates that transform the class of underactuated systems into a normal form [1]. Specific formulations for the different types of configurations are shown in [2]. Hence, for the given set of shape variables q^m, we can choose the zero coordinates to be $z_1 := q^z, z_2 := D_z \dot{q}$, where D_z consists of the rows of D that correspond to the unactuated q_i's. For example, if q^m is fully actuated, then $D_z = [D_{11}\ D_{12}]$. The dynamics of z can thus be derived accordingly. With this property, we have the following result.

Theorem 2: *Given the class of manipulators $\mathcal{BD}$, let the configuration set Q_u, and the desired configuration q_d be picked such that Assumptions 1 and 2 are satisfied. In addition, let the system satisfy the kinetic symmetry property w.r.t. the external variables q^z. If the system belongs to one of the following categories:*

(a) *The shape variables q^m are actuated,*

(b) *The shape variables q^m are unactuated/partially actuated, $D_{11}(q^m)$ has constant terms, and $D_{21}(q^m)$ satisfies the differential conditions (Remark 2),*

then for every $T > 0$, and for every $(q_0, \dot{q}_0) \in T\mathrm{Q}_u$, there exist sufficiently large enough gains $k_p, k_d > 1$ such that the outputs $(e(t), \dot{e}(t))$ are exponentially convergent to a (desirable) bound for all $t \in [0, T]$.

Remark 2: A matrix function $M : \mathbb{R}^m \to \mathbb{R}^{m \times l}$ is said to satisfy the differential conditions if i^{th} row of the matrix M (denoted by M_i) satisfies $\frac{\partial M(q^m)^T}{\partial q_i^m} = \frac{\partial M_i(q^m)^T}{\partial q^m}$, for $i = 1, 2, \dots, m$. Here, q_i^m denotes the i^{th} element of q^m. It is worth noting that this is a generalized version of the differential symmetric conditions shown in [2, Definition 4.2.2] for square matrices. As an example, cart-pole systems satisfy these conditions. With this definition, we now prove Theorem 2. It is divided into two parts.

Proof: [Proof of Theorem 2(a)] In the interest of space, we will follow the steps in [13, Proofs of Lemma 1 and 2], and then highlight the changes that establish Theorem 2(a). We will first establish boundedness for a small enough interval $[0, T_\delta]$, and then stretch T_δ to T. To establish convergence to a bound, we consider the following Lyapunov candidate:

$$V_e(e, \dot{e}, q) = V_0(e, \dot{e}, q) + V_c(e, \dot{e}, q) \quad (13)$$

$$V_0(e, \dot{e}, q) = \frac{1}{2} \begin{bmatrix} e \\ \dot{e} \end{bmatrix}^T \begin{bmatrix} \iota K_p & \mathbf{0} \\ \mathbf{0} & D_u(q) \end{bmatrix} \begin{bmatrix} e \\ \dot{e} \end{bmatrix} \quad (14)$$

$$V_c(e, \dot{e}, q) = \alpha(e) e^T D_u(q) \dot{e} \quad (15)$$

$$\alpha(e) = \frac{k_0}{1 + |e|} = \frac{k_0}{1 + \sqrt{e^T e}}. \quad (16)$$

Here V_e is similar to the Lyapunov function chosen in [13, (54)], except for the inclusion of the constant term ι. $k_0 > 0$ is chosen such that V_e is positive definite. Accordingly, we have the following bounds on V_e:

$$\lambda_{\mathrm{l}} \left| \begin{bmatrix} \sqrt{k_p} e \\ \dot{e} \end{bmatrix} \right|^2 \le V_e \le \lambda_{\mathrm{u}} \left| \begin{bmatrix} \sqrt{k_p} e \\ \dot{e} \end{bmatrix} \right|^2, \quad (17)$$

for some positive constants λ_{l}, λ_{u} that do not depend on k_p.

Following the steps similar to [13, (62)-(65)], we have the derivative of V_e as

$$\begin{aligned} \dot{V}_e \le & -\frac{\alpha}{2} \begin{bmatrix} e \\ \dot{e} \end{bmatrix}^T \begin{bmatrix} k_p(B_u + B_u^T) & \frac{k_p(B_u^T - \iota \mathbf{1}) + \alpha k_d B_u}{\alpha} \\ \frac{k_p(B_u - \iota \mathbf{1}) + \alpha k_d B_u^T}{\alpha} & \frac{k_d(B_u + B_u^T)}{\alpha} \end{bmatrix} \begin{bmatrix} e \\ \dot{e} \end{bmatrix} \\ & + c_u(\alpha|e| + |\dot{e}|)(|z_2|^2 + 1) \\ & + 2c_u(\alpha + \alpha|e| + |\dot{e}| + 1)|\dot{e}|^2, \end{aligned} \quad (18)$$

where the constant $c_u > 0$ is simply redefined to collect all the constant terms. It can be verified that by kinetic symmetry, $\dot{D}_u$ is only dependent on $q^m, \dot{q}^m$. Therefore, only the second summand depends on z_2 (compared to [13, (65)]). Similarly, following the steps in [13, (65)-(73)], we choose the control gains k_p, k_d in such a way that V_e is decreasing. Therefore we choose $k_p = \varepsilon^2$ and $k_d = k\varepsilon$ for some $k, \varepsilon > 1$. For $|z_2|$, we will pick a tube of radius r (say) around $z^*(t)$, and let T_δ be the time when z crosses this radius. In this compact tube around $z^*(t)$, where $t \in [0, T_\delta]$, let the maximum value of $z(t)$ be b. We have the following inequality:

$$\dot{V}_e \le -\frac{\alpha \lambda_{\min}(\Lambda)}{4} |(\varepsilon e, k\dot{e})|^2 + k_1(\alpha|e| + |\dot{e}|), \quad (19)$$

where $k_1 = c_u(|z_2|^2 + 1)$, and Λ is given by Assumption (2) (which is only dependent on q^m, and has a minimum eigenvalue in a compact tube in Q_u specified by the interval $[0, T]$ and the initial states). The above inequality is satisfied as long as z remains in the tube. Accordingly, we have that

$$V_e(t) \le e^{-2\varepsilon\lambda t} V_e(0) + \frac{k_b}{\varepsilon^2}, \quad t \in [0, T_\delta], \quad (20)$$

where λ, k_b are obtained by collecting all the additional terms that are independent of ε (see [13, (69)-(72)]). k_b is shown with its subscript to indicate its dependence on b. We can express the above inequality in terms of the outputs as

$$\left| \begin{bmatrix} e(t) \\ \dot{e}(t) \end{bmatrix} \right| \le \varepsilon e^{-\varepsilon \lambda t} \sqrt{\frac{\lambda_{\mathrm{u}}}{\lambda_{\mathrm{l}}}} \left| \begin{bmatrix} e(0) \\ \dot{e}(0) \end{bmatrix} \right| + \sqrt{\frac{k_b}{\varepsilon^2 \lambda_{\mathrm{l}}}}, \quad (21)$$

where (17) is substituted.

Having established convergence in $[0, T_\delta]$, we will now stretch T_δ to T. Since q^m is actuated, we know from (7) that the dynamics of z_2 will only consist of terms dependent on q, and not on the velocity $\dot{q}$. Therefore, we have the following:

$$\begin{aligned} |z(t) - z^*(t)| & \le \int_0^t |\dot{z}(s) - \dot{z}^*(s)| \, ds \\ & \le c_z \int_0^t |\eta(s)| ds + c_z \int_0^t |z(s) - z^*(s)| ds, \end{aligned}$$

for some $c_z > 0$. Here $\eta(t) := (e(t), \dot{e}(t))$ for convenience. In comparison, [13, (78)] has terms quadratic in $|\eta(s)|$ and $|z(s) - z^*(s)|$. Also note that $z(0) = z^*(0)$. By using Gronwall-Bellman inequality [20, Lemma A.1], we obtain

$$|z(t) - z^*(t)| \le \left(\frac{c_z}{\lambda} \sqrt{\frac{\lambda_{\mathrm{u}}}{\lambda_{\mathrm{l}}}} |\eta(0)| + c_z \sqrt{\frac{k_b}{\varepsilon^2 \lambda_{\mathrm{l}}}} T_\delta \right) e^{c_z t}.$$

This shows that the maximum possible value above decreases with increasing ε. Therefore, given $\eta(0)$ and T, we can set a suitable upper bound b for the specified interval, and then increase ε such that $z(t)$ still remains within the tube in $[0, T]$. This ensures that convergence of $\eta(t) = (e(t), \dot{e}(t))$ is achieved to desirable values. This completes the proof. ■

Proof: [Proof of Theorem 2(b)] Proof of this part is straightforward after proving part (a). We first note that the dynamics obtained from (2) will contain velocity dependent terms like $\dot{D}(q^m, \dot{q}^m)\dot{q}$ and $\partial \dot{q}^T D(q^m)\dot{q} / \partial q$. Remaining terms will not contain $\dot{q}^z, \dot{q}^m$. Since $\dot{D}$ is only dependent on $q^m, \dot{q}^m$ and D_{11} is a constant, we have $\dot{D}_{11} = 0$, and $\partial(\dot{q}^{zT} D_{11} \dot{q}^z)/\partial q = 0$. In addition, by differential conditions, row i of D_{21} satisfies

$$\dot{D}_{21_i}(q^m, \dot{q}^m)\dot{q}^z = \frac{\partial}{\partial q_i^m} \dot{q}^{mT} D_{21}(q^m) \dot{q}^z. \quad (22)$$

It can be verified that the resulting dynamics obtained from (2) will not contain terms dependent on $\dot{q}^z$. This implies that the resulting derivative of the Lyapunov function (18) will not be containing terms dependent on z_2. Hence, the convergence result follows directly. ■

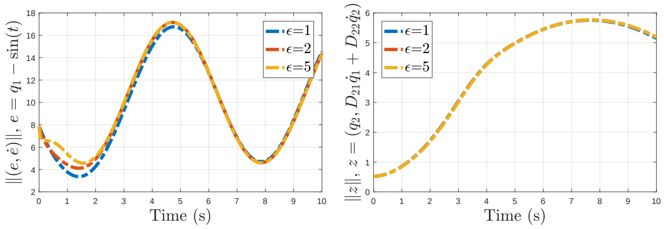

Fig. 4. Figure showing the tracking results for the acrobot. The error is expressed in radians.

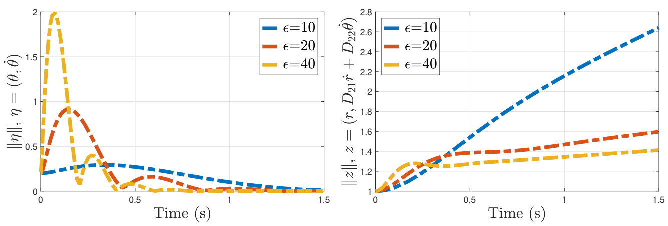

Fig. 5. Figure showing the tracking results for the cart-pole system. Error is expressed in radians.

V. Simulation results

In this section we will briefly discuss the simulation results for two robot models:

1) Acrobot: Both the links are having mass $m = 0.05$ kg. Link length is $l = 1$ m, and $l_c = 0.1$ m (see Fig. 1). The states are $(q_1, q_2, \dot{q}_1, \dot{q}_2)$. Only q_1 is actuated. The goal is to have q_1 track a sinusoidal trajectory. Therefore, this system satisfies the conditions of Theorem 2(a). Results are shown in Fig. 4. The gains used are $k_p = \varepsilon^2$ and $k_d = 2\varepsilon$, with $\varepsilon = 1, 2, 5$. Increasing ε results in smaller bounds. Note that q_2 will be floating since the PD control does not provide convergence guarantees for the uncontrolled states of the system.

2) Cart-pole: Mass of the cart is $m_1 = 5$ kg, and that of the pole is $m_2 = 1$ kg. Length is $\ell = 1$ m (see Fig. 1). r is the cart position, and θ is the pole angle. Since the kinetic energy is symmetric w.r.t. the cart-position, with D_{11} being a constant, and D_{21} satisfying the differential conditions, we can apply Theorem 2(b). Note that these conditions were also used in [2, Proposition 4.2.1] for cart-pole systems. The goal is to drive $\theta \to 0$, and the results are shown in Fig. 5. The gains used are $k_p = \varepsilon^2$ and $k_d = 2\varepsilon$, with $\varepsilon = 10, 20, 40$. It can be verified that the zero coordinates are increasing in magnitude w.r.t. time, whereas θ is exponentially decreasing.

VI. Conclusions

PD based control laws are known to exhibit low sensitivity to modeling errors, and are very easy to implement for all kinds of trajectory tracking applications. Hence, in this letter, we showed that some of the stability guarantees existing for fully actuated systems can still be extended for underactuated robotic systems. For the class of $\mathcal{BD}$ manipulators with non-interacting inputs, and for desired trajectories with bounded derivatives, we can tune PD gains to yield local convergence and boundedness guarantees. In addition, for a sub-class of robotic systems containing kinetic symmetry, stronger convergence guarantees can be provided. Future work will involve including noise, torque saturations, and establishing stability guarantees for a broader class of underactuated robotic systems.

References

[1] A. Isidori, "Nonlinear control systems," 1997.

[2] R. Olfati-Saber, "Nonlinear control of underactuated mechanical systems with application to robotics and aerospace vehicles," Ph.D. dissertation, Massachusetts Institute of Technology, 2001.

[3] M. W. Spong, "Partial feedback linearization of underactuated mechanical systems," in *Proceedings of IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS'94)*, vol. 1, Sep. 1994, pp. 314–321 vol.1.

[4] A. R. Teel, "Using saturation to stabilize a class of single-input partially linear composite systems," in *Nonlinear Control Systems Design 1992*. Elsevier, 1993, pp. 379–384.

[5] K. J. Åström and K. Furuta, "Swinging up a pendulum by energy control," *Automatica*, vol. 36, no. 2, pp. 287–295, 2000.

[6] T. Samad, S. Mastellone, P. Goupil, A. van Delft, A. Serbezov, and K. Brooks, "Ifac industry committee update, initiative to increase industrial participation in the control community," in *Newsletters April 2019*. IFAC, 2019.

[7] M. Takegaki and S. Arimoto, "A New Feedback Method for Dynamic Control of Manipulators," *Journal of Dynamic Systems Measurement and Control-transactions of The Asme*, vol. 103, 1981.

[8] S. Arimoto, "Stability and robustness of PID feedback control for robot manipulators of sensory capability," *International Journal of Robotics Research*, pp. 783–799, 1984.

[9] D. E. Koditschek, "Strict global lyapunov functions for mechanical systems," in *Proceedings of American Control Conference*, June 1988, pp. 1770–1775.

[10] ——, "Robot planning and control via potential functions," pp. 349–367, 1989.

[11] L. L. Whitcomb, A. A. Rizzi, and D. E. Koditschek, "Comparative experiments with a new adaptive controller for robot arms," *IEEE Transactions on Robotics and Automation*, vol. 9, no. 1, pp. 59–70, Feb 1993.

[12] S. Mori, H. Nishihara, and K. Furuta, "Control of unstable mechanical system control of pendulum," *International Journal of Control*, vol. 23, no. 5, pp. 673–692, 1976.

[13] S. Kolathaya, "Local stability of PD controlled bipedal walking robots," *Automatica*, vol. 114, p. 108841, 2020. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S000510982030039X

[14] F. Ghorbel, B. Srinivasan, and M. W. Spong, "On the uniform boundedness of the inertia matrix of serial robot manipulators," *Journal of Robotic Systems*, vol. 15, no. 1, pp. 17–28, 1998. [Online]. Available: http://dx.doi.org/10.1002/(SICI)1097-4563(199812)15:1¡17::AID-ROB2¿3.0.CO;2-V

[15] S. N. Yadukumar, M. Pasupuleti, and A. D. Ames, *From Formal Methods to Algorithmic Implementation of Human Inspired Control on Bipedal Robots*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2013, pp. 511–526.

[16] C. Urrea, J. Corts, and J. Pascal, "Design, construction and control of a scara manipulator with 6 degrees of freedom," *Journal of Applied Research and Technology*, vol. 14, no. 6, pp. 396 – 404, 2016. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S1665642316300931

[17] R. Gunawardana and F. Ghorbel, "On the uniform boundedness of the coriolis/centrifugal terms in the robot equations of motion," *International Journal of Robotics and Automation*, vol. 14, no. 2, pp. 45–53, 1999.

[18] R. Gunawardana and F. Ghorbel, "The class of robot manipulators with bounded jacobian of the gravity vector," in *Proceedings of IEEE International Conference on Robotics and Automation*, vol. 4, April 1996, pp. 3677–3682 vol.4.

[19] E. Minguzzi, "Completeness of first and second order ode flows and of eulerlagrange equations," *Journal of Geometry and Physics*, vol. 97, pp. 156 – 165, 2015. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0393044015001382

[20] H. K. Khalil, *Nonlinear systems*. Prentice hall Upper Saddle River, NJ, 2002, vol. 3.